YOUNG MATHEMATICIANS AT WORK

Constructing Algebra

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1 | ALGEBRA: STRUCTURES OR STRUCTURING?

When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again. The never-satisfied man is so strange, for he completes a structure not in order to dwell in it peacefully, but in order to begin another.

—*Karl Friedrich Gauss, Letter to Bolyai (1808)*

Nothing is more important than to see the sources of invention, which are in my opinion more interesting than the inventions themselves.

—*G. W. Leibniz (1646–1716)*

What is algebra? How does it develop? Because algebra includes many things—generalizing beyond specific instances, describing and representing patterns and functions, building equations and expressions using symbolic representations with integers and variables, manipulating symbols to solve for unknowns—there has been a spirited debate as researchers have tried to define the topic, specifically as it relates to teaching this strand of mathematics in elementary and middle school.

Some researchers have argued that algebra in the elementary school should be thought of as the construction of algebraic "big ideas" growing out of generalized arithmetic (Schifter, Russell, and Bastable 2006; Carpenter, Franke, and Levi 2003). Others (Driscoll 1999) describe algebra as a type of reasoning in which one investigates the relationships between specific cases and possible generalizations and develops algebraic "habits of mind"—ways of thinking about algebraic questions. Whereas early work on algebra in the sixties' "new math" movement focused on examining algebraic structures (Wirtz, cited in Goldenburg and Shteingold 2008), more recent work has examined the process of "algebrafying" (for example, systematically symbolizing generalizations) (Kaput, Carraher, and Blanton 2008).

Over the years, our own participation in this discussion was not directed toward defining algebra as much as it was toward studying its emergence. Our questions have been: What might the development of algebra look like in the elementary grades? What are some of the critical big ideas and strategies young children construct that might serve as important landmarks for teachers to notice, develop, and celebrate? What causes some of the misconceptions and challenges that develop? How might realistic contexts and representational models—a double number line, combination charts, and the ratio table, for example—support the development of algebra? As we worked in classrooms attempting to answer these questions, our own working definition of early algebra began to emerge.

TEACHING AND LEARNING IN THE ALGEBRA CLASSROOM

It's a beautiful crisp fall day in New York City. Bill is leading a Math in the City (MitC) professional development session for twelve elementary teachers. The group has enthusiastically agreed to come together for four days over the course of the semester to deepen their own understanding of algebra and to study how the topic might be taught in the elementary grades. In a typical session the participants explore and discuss a mathematical investigation, choose an inherent idea, and craft a context in which to explore this idea in a fifth-grade classroom in the host school. Then one of the participants teaches the lesson while the others observe. Finally everyone reflects on the teaching and learning that has taken place. Today Bill is focusing on algebraic structures in the number system.

"How many factors does the number 1 have?" Bill begins.

The question is easy. "Just one," several teachers exclaim immediately.

"And the numbers 2 and 3 each have two. What about the number 4?"

"That has three," Camille, a seasoned fourth-grade teacher, replies quickly. "1, 2, and 4 are all factors of 4."

"And the number 6 has four factors, right, 1, 2, 3, and 6? Okay. So let's investigate this. Work with a partner and sort numbers by the number of factors they have. Let's see if we can find any interesting relationships."

Camille and Karen, working with one number at a time, generate a table to show their results (see Figure 1.1). When they reach 13, they begin to discuss what they have noticed.

"Thirteen's a prime number. All the primes are going to have just two factors," Camille says, filling in the chart's cells for 13. Then she whispers, "You know, this is embarrassing; I knew 1 wasn't a prime number because a teacher told me that once and I accepted it. But my definition of primes was a number that had factors of 1 and itself only. I never quite understood why 1 wasn't a prime because it fit my definition, but in our list I see now why it isn't. It's by itself. All the primes are going to have two factors. The number 1 only has one factor."

Karen nods. "I think I'm seeing another pattern. It seems that if we double a prime number, we have to add two to the number of factors. See, 3 is a prime number, right? And it has two factors. If we double 3, we get 6, and that number has four factors. The number 5 has two factors, and 10 has four. I think 14 will have four factors, too. Yep—1, 7, 2, and 14."

Camille is intrigued. "That's interesting." She ponders a minute and then realizes what is happening. "Oh, I get it. We are adding two more factors. First it was 1 and 7, but when we doubled the number, the new product and the number 2 became factors, too. That should always work."

"Yeah, that makes sense, but why didn't it work with the number 2? The number 4 has three factors, not four." She thinks a moment. "Maybe because 2 was already a factor? Hey, maybe odds and evens have something to do with it?" *Sometimes finding an example that disproves a conjecture can prompt a new insight.* "Hey, the rest of the prime numbers are odd! I just realized . . . of course, they have to be, because to be even 2 would have to be a factor!"

Delighted, Karen writes on their chart, "All primes except 2 are odd and when you double an odd prime you get two more factors—so plus two."

Meanwhile, across the room George and Maria have created a horizontal chart showing the number of factors as numbers increase by one (see Figure 1.2).

George, chunking the numbers into overlapping triads, notes a pattern. "This is interesting. I see a 232, then a 242, but then a 243 and a 342. The numbers are always larger in the middle."

"Yes, I see that but I don't see what the pattern really is. I mean I don't think we can use it to predict it."

Number	Factors	Number of Factors
1	1	1
2	1, 2	2
3	1, 3	2
4	1, 2, 4	3
5	1, 5	2
6	1, 2, 3, 6	4
7	1, 7	2
8	1, 2, 4, 8	4
9	1, 3, 9	3
10	1, 2, 5, 10	4
11	1, 11	2
12	1, 2, 3, 4, 6, 12	6
13		

FIGURE 1.1
Camille and Karen's Table

Bill has been listening to this conversation and joins the pair. "George, are you thinking about going from one number to the next?"

FIGURE 1.2
George and Maria's Chart

1	2	3	4	5	6	7	8	9	10	11
1	2	2	3	2	4	2	4	3	4	2

"Yes, isn't that what you do when you search for patterns? You asked us to look for patterns, right?"

"Well, I didn't exactly ask you to search for patterns. I think I said something about relationships you might notice when you sort numbers by their number of factors." Maria still looks puzzled, so Bill attempts to reframe the investigation. "Try thinking about this. Factors depend upon multiplication, right? Maybe it would help to sort the numbers into piles with the same number of factors and think about how they are related in terms of multiplication."

Maria begins to formulate a new direction for their inquiry. "You mean like, six is two times three and ten is two times five? They are both double odds."

"That's an interesting idea. Keep going and I'll check back with you in a bit."

In many classrooms algebra is taught as recognizing and extending patterns, and George and Maria initially try to do that by looking for sequential patterns. While certainly a part of early algebra, this notion is not sufficiently general. Bill has deliberately chosen an investigation in which sequential (or additive) structuring will not make sense. Numbers can be structured in many ways: in a sequential (plus one) order, additively, multiplicatively, exponentially, and so on. The paradigmatic shift from additive to multiplicative structuring took centuries!

Bill returns to Camille and Karen to see whether they are making progress with their "plus two" idea when they double an odd prime. Camille begins generalizing her observation. "Hey, maybe it's not just primes! Maybe it's all odd numbers. Nine has three factors: 1, 3, and 9 . . . oh no, 18 is the double but it has six factors: 1, 18, 2, 9, 3, and 6. If it had been plus two, it should have had only five factors." She lays out the original pairs for 9 and then compares them with the pairs for 18 (see Figure 1.3).

FIGURE 1.3
Camille's Organization of Factors of 18

1, 9	1, 18
3	2, 9
	3, 6
Factors of 9	Factors of 18